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Similar Solutions for Merging Shear Flows II

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IN Ref. 1 the merging of two infinite uniform shear flows behind a trailing edge was investigated, including the possible effects of the so-called "vorticity-induced" pressure gradient. The present note summarizes the results of subsequent numerical calculations. Detailed results covering a wide range of initial vorticity ratios of the two infinite shear flows are available in Ref. 2.

The notation of Ref. 1 is used. The plane incompressible flows are separated by a plane from $x = -\infty$ to $x = 0$; the initial vorticities are Ω_1 and $-\Omega_2$ in the upper and lower half-plane, respectively. The (laminar) merging takes place at $x > 0$. The similarity properties of the merging layer are accounted for by setting the stream function ψ equal to

$$\psi = (\Omega_1 \nu^2 x^2)^{1/3} f(\eta) \qquad (1)$$

where

$$\eta = y(\Omega_1/\nu x)^{1/3} \qquad (2)$$

as was first proposed by Goldstein.[3] Rott and Hakkinen[1] admitted the possibility of a vorticity-induced pressure gradient that was found to have the form

$$dp/dx = \rho c \Omega_1^{4/3} \nu^{2/3} x^{-1/3} \qquad (3)$$

where c is a dimensionless parameter to be discussed later.

These assumptions together with the standard boundary-layer formulation of the Navier-Stokes equations yield the total differential equation

$$3f''' + 2ff'' - f'^2 = 3c \qquad (4)$$

with boundary conditions $f'' = 1$ at $\eta = \infty$, and $f'' = -\Omega_2/\Omega_1 = -\beta$ at $\eta = -\infty$. The following behavior is then implied for the asymptotic velocity profiles as $\eta \rightarrow \pm\infty$:

$$f'_{+\infty} = \eta - l_1 \qquad f'_{-\infty} = -\beta(\eta - l_2) \qquad (5)$$

where l_1 and l_2 can be determined only after numerical integration of Eq. (4).

There are two families of solutions for which numerical calculations were performed. The first case is characterized by $c = 0$, i.e., vanishing pressure gradient, and the second one by unshifted asymptotic velocity profiles, i.e., $l_1 = l_2 = 0$, corresponding to a specific induced pressure gradient parameter $c \neq 0$. In terms of physical velocities, the general asymptotic behavior is

$$u_{+\infty} = \Omega_1[y - \Omega_1^{-1/3} (\nu x)^{1/3} l_1]$$
$$u_{-\infty} = \Omega_2[y - \Omega_1^{-1/3} (\nu x)^{1/3} l_2] \qquad (6)$$

To obtain the special solution with $l_1 = l_2 = 0$, the parameter c has to be varied until $l_1 - l_2 = 0$. Noting that Eq. (4) is invariant with respect to a change in the origin of η, it is then possible to find a position where the asymptotic u-distributions remain unshifted both for $y = +\infty$ and $y = -\infty$.

The order of magnitude of the shift as seen from Eq. (6) (noting that l_1 and l_2 are of order 1) gets properly small in the limit of vanishing kinematic viscosities but shows an undue growth for large x. The significance of this behavior is hard to ascertain, as the undisturbed velocity also grows without bounds for large $|\eta|$. Nevertheless, in a similar situation involving a flat-plate boundary layer in an infinite shear flow, Li[4] has first proposed that solutions involving a shift at infinity are not permissible.

The investigation of Murray[5] upheld Li's contention, provided that the shear flow is indeed extending to infinity laterally. In what limited sense only can a laterally bounded shear flow region be approximated by the infinite case has been shown by Toomre and Rott.[6] Infinite shear flow turned out to be a limit of questionable properties, both mathematically and physically. Only for a region whose extent is small compared to the lateral dimension of the oncoming shear flow can the Li-Murray solution be applied.

Similar investigations for the problem of merging shear flows are still incomplete. Here, the Li-Murray boundary conditions are applied without further discussion to the numerical calculation of the symmetric case, i.e., $\Omega_1 = -\Omega_2$, $l_1 = l_2 = 0$, and the corresponding value of the pressure gradient parameter turns out to be $c = 0.4089$. The remaining discussion is concerned with the possible significance of this solution for the merging of two identical Blasius boundary layers at the sharp trailing edge of a flat plate. The vorticity there, for a plate of the length L, is given by the Blasius solution and has the value

$$\Omega_1 = -\Omega_2 = \Omega = 0.332\, U_\infty^{3/2}(\nu L)^{-1/2} \qquad (7)$$

A quantity of special interest is the velocity at the line of symmetry $\eta = 0$:

$$u_0/U_\infty = 0.4795\, f'(0)(x/L)^{1/3} \qquad (8)$$

that is, for case 1, $c = 0$, Goldstein's leading term:

$$u_0/U_\infty = 0.772(x/L)^{1/3}$$

and for case 2, $l_1 = l_2 = 0$, present solution

$$u_0/U_\infty = 0.431(x/L)^{1/3}$$

How far downstream the new solution may be valid is not known precisely, but the analogy with the forementioned results of Toomre and Rott[6] permits a qualitative statement. Noticeable influence of the vorticity-induced pressure gradient can be found only in a region with streamwise dimension comparable to the lateral extent of the shear layer. The latter quantity is, for the present application, equal to the Blasius boundary-layer thickness:

$$\delta_{Bl}/L \cong 5(\nu/U_\infty L)^{1/2} = 5Re_L^{1/2} \qquad (9)$$

There exists a second limitation on the validity of the solution considered here; for regions too near to the trailing edge, the boundary-layer assumptions have to be abandoned in favor of new approximations of the Stokes or Oseen type.

For regions far from the trailing edge, an asymptotic expansion procedure, in which the similarity solution appears as the leading member, has been carried out by Hakkinen and O'Neil.[7] These expansions all diverge as the trailing edge is approached.

The main results of Hakkinen and O'Neil[7] can be summarized as follows: 1) the downstream centerline velocity is given by the series:

$$u_0 = \Omega^{2/3}\nu^{1/3}c^{1/3}(0.899 - 0.165[x(\Omega/\nu)^{1/2}]^{-2/3} - 0.885[x(\Omega/\nu)^{1/2}]^{-4/3} + \ldots) \qquad (10)$$

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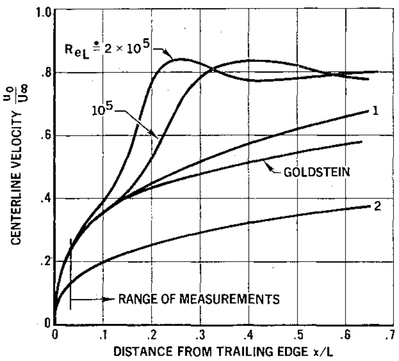

Fig. 1 Experiment of Sato and Kuriki (plate I, Ref. 9) as compared with Goldstein's result,[10] the corresponding similarity solution (case 1, $c = 0$), and the present similarity solution (case 2, $l_1 = l_2 = 0$).

and, 2) the upstream effect on the wall shear by

$$\tau/\tau_{-\infty} = 1 + 0.254\ [|x|(\Omega/\nu)^{1/2}]^{-2/3} - 0.004[|x|(\Omega/\nu)^{1/2}]^{-4/3} + \ldots \quad (11)$$

It is interesting to note that there is apparently a sizable range of $|x|(\Omega/\nu)^{1/2}$ where τ/τ_∞ turns out to be larger than 1. The reason is that the fluid entering the wake is accelerated, and the corresponding "displacement effect" (caused by the "negative displacement thickness" of the wake) increases the upstream velocity and the shear. In order to estimate the region of validity of the similarity solution, the first term in the asymptotic expansion (10), one sees that for

$$[|x|_{\lim}(\Omega/\nu)^{1/2}]^{-2/3} = 0.4 \quad (12)$$

the known consecutive terms in Eq. (11) decrease by at least a factor of 0.1, and the terms in Eq. (10) also decrease reasonably fast. It is also reassuring to note that the form of (12) is compatible with an outer limit of validity criterion based on Imai's proposed theory for flow in the vicinity of a trailing edge.[8]

Now, the validity of the theory presented in this paper, including the effect of the vorticity-induced pressure gradient, supposes x to be less than δ_{Bl} from Eq. (9) but larger than $|x|_{\lim}$ after Eq. (12), i.e.,

$$|x|_{\lim}/L = 7Re_L^{-3/4} < x/L < 5Re_L^{-1/2} \cong \delta_{Bl}/L \quad (13)$$

For

$$Re_L = 10^2,\ 0.48 < x/L < 0.50$$

$$Re_L = 3 \times 10^3,\ 0.018 < x/L < 0.09$$

$$Re_L = 10^5,\ 0.001 < x/L < 0.015$$

It is seen that the Reynolds number must be sufficiently large to provide a separation between the upper and lower limits or validity; on the other hand, it should remain well below the laminar stability limit. In this range of Reynolds numbers, the vorticity-induced pressure gradient should then be observable in a small region near the trailing edge. Whether or not actual measurements are indeed possible depends, of course, on the size of the experimental apparatus relative to the physical extent of the region under consideration.

At this time, the only applicable experimental investigation of the merging of two Blasius boundary layers is that carried out by Sato and Kuriki.[9] Reynolds number range extended from approximately 10^5 to 2×10^5; in this range, the possible existence of the vorticity-induced pressure gradient region can be anticipated. According to Eq. (9), the boundary-layer thickness δ_{Bl}/L varied from 0.015 to 0.01, whereas $|x|_{\lim}/L$ was bracketed by the values 0.001 and 0.0006.

Figure 1 shows the measurements of u_0/U_∞ by Sato and Kuriki,[9] together with theoretical curves after Goldstein,[3,10] and with the inclusion of the vorticity-induced pressure gradient effect (curve 2). The points measured by Sato and Kuriki begin at x = 10mm; this gives, with a plate length L = 3000 mm, $x/L = 3 \times 10^{-2}$, which lies outside of the discernible pressure gradient effect region for the Reynolds numbers of the test. Indeed, the measurements of Sato and Kuriki follow Goldstein's curve closely, before the effect of instability takes over and leads to major deviations from curve 1.

Although the measurements of Sato and Kuriki do not penetrate into the predicted region of validity for curve 2, Fig. 1 nevertheless illustrates the difficulties of an experiment that is designed to discern between curves 1 and 2 in that region. At the present time, experimental evidence concerning the existence of an induced pressure gradient in merging shear flows must be considered inconclusive.

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Shock Predictions in Conical Nozzles

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Introduction

PREVIOUS analyses indicate that conventional conical nozzles are not shock free[1,2] and that the initial shock formation occurs near the axis of symmetry.[2] There are several applications where the formation of a shock in this region is of importance. For example, conventional conical nozzles are often used to study nonequilibrium flow,[3,4] and there exists the possibility of obscuring the chemical effects with aerodynamic factors. In the study of flow over bodies placed along the centerline of conical wind-tunnel nozzles, the freestream conditions cannot be properly assessed without a knowledge of the effects of shock formation; and for the same reason, the design of contoured wind-tunnel nozzles

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